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GENERALIZATION OF INTEGRAL THEOREM REPRESENTATIONS

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Summary. It is shown that the known theorems about volume integration of some differential-vectorial operations can be written as one analytical expression.

Key words: volume integration, surface integration.

INTRODUCTION

Nowadays the theorems about volume integration of differential-vectorial operations (divergence, rotor, gradient), that allow to bring it to the corresponding surface integration, are known [1-3]. The Green theorem [1-3] also gives a possibility to bring volume integration of differential-vectorial operations with two scalar functions to the corresponding surface integration. All these theorems have different forms of representation. However, all these forms have something general in the structure [3].

In this paper it is shown, that allthe above-mentioned theorems can be generalized in the following form:

$$\int_V \left(\sum_{\zeta=x,y,z} \frac{\partial \psi(\bar{a},\bar{b},\zeta)}{\partial \zeta} \right) \cdot dV = \int_S \psi(\bar{a},\bar{b},\bar{n}) \cdot dS \,, \tag{1}$$

where: Ψ – function, defined in V, that represents certain algebraic combination of vectors $\bar{a}$, $\bar{b}$ and normal vector $\bar{n}$ to the surface S.

1. THEOREM ABOUT VOLUME INTEGRATION OF DIVERGENCE OF VECTOR

Let Ψ be the scalar multiplication of vectors $\bar{a}$ and $\bar{n}$:

$$\Psi = \bar{a} \cdot \bar{n} \,. \tag{2}$$

For such type of the function Ψ the following equations can be written:

$$\frac{\partial\psi(\bar{a},x)}{\partial x}=\frac{\partial}{\partial x}(\bar{a}\cdot\bar{1}_x)=\frac{\partial a_x}{\partial x},$$
$$\frac{\partial\psi(\bar{a},y)}{\partial y}=\frac{\partial}{\partial y}(\bar{a}\cdot\bar{1}_y)=\frac{\partial a_y}{\partial y},$$
$$\frac{\partial\psi(\bar{a},z)}{\partial z}=\frac{\partial}{\partial z}(\bar{a}\cdot\bar{1}_z)=\frac{\partial a_z}{\partial z},$$

that allow to write the equation (1) in the form:

$$\oint_S(\bar{a}\cdot\bar{n})\cdot dS=\oint_V\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)\cdot dV=\oint_v \mathrm{div}\bar{a}\cdot dV,$$

which is known as the Ostrogradskiy-Gauss theorem [1, 2].

2. THEOREM ABOUT VOLUME INTEGRATION OF ROTOR OF VECTOR

Let Ψ be vectorial multiplication of vectors $\bar{a}$ and $\bar{n}$:

$$\Psi=\bar{a}\times\bar{n}. \tag{3}$$

For such a type of the function Ψ the following equations can be obtained by factoring of vectorial multiplication $\bar{a}\times\bar{1}\zeta$:

$$\frac{\partial\Psi(\bar{a},x)}{\partial x}=\frac{\partial}{\partial x}([\bar{a}\times\bar{1}_x])=\frac{\partial}{\partial x}\left(\bar{a}_y\cdot\bar{1}_z-\bar{a}_z\cdot\bar{1}_y\right), \tag{4}$$
$$\frac{\partial\Psi(\bar{a},y)}{\partial y}=\frac{\partial}{\partial y}([\bar{a}\times\bar{1}_y])=\frac{\partial}{\partial y}\left(\bar{a}_z\cdot\bar{1}_x-\bar{a}_x\cdot\bar{1}_z\right),$$
$$\frac{\partial\Psi(\bar{a},z)}{\partial z}=\frac{\partial}{\partial z}([\bar{a}\times\bar{1}_z])=\frac{\partial}{\partial z}\left(\bar{a}_x\cdot\bar{1}_y-\bar{a}_y\cdot\bar{1}_x\right).$$

As a result, taking into account (4) and the analytical form of representation of rotor of vector in cartesian coordinates [1], it is possible to write the next correlation:

$$\sum_{\zeta=x,y,z}\frac{\partial}{\partial\zeta}\left(\psi\left(\bar{a},\bar{1}_\zeta\right)\right)=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\cdot\bar{1}_x+$$
$$+\left(\frac{\partial a_x}{\partial z}-\frac{\partial az}{\partial x}\right)\cdot\bar{1}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial ax}{\partial y}\right)\cdot\bar{1}=\mathrm{rot}\bar{a},$$

that in this case allows to present (1) in the following way:

$$\oint_S\bar{a}\times\bar{n}\cdot dS=\oint_v \mathrm{rot}\bar{a}\cdot dV,$$

which is the notation of the known theorem about volume integration of rotor of vector [1, 2].

3. THEOREM ABOUT VOLUME INTEGRATION OF GRADIENT OF SCALAR FUNCTION

Let the function Ψ be multiplication of vector $\bar{n}$ and scalar function φ:

$$\Psi = \bar{n} \cdot \varphi . \tag{5}$$

For such kind of the function Ψ it is possible to write the following equations:

$$\frac{\partial \psi(\bar{a}, \bar{b}, x)}{\partial x} = \frac{\partial(\varphi \cdot \bar{1}_x)}{\partial x} = \frac{\partial \varphi}{\partial x} \cdot \bar{1}_x ,$$
$$\frac{\partial \psi(\bar{a}, \bar{b}, y)}{\partial y} = \frac{\partial(\varphi \cdot \bar{1}_y)}{\partial y} = \frac{\partial \varphi}{\partial y} \cdot \bar{1}_y ,$$
$$\frac{\partial \psi(\bar{a}, \bar{b}, z)}{\partial Z} = \frac{\partial(\varphi \cdot \bar{1}_z)}{\partial Z} = \frac{\partial \varphi}{\partial Z} \cdot \bar{1}_z ,$$

which results in:

$$\sum_{\xi = x,y,z} \left(\frac{\partial}{\partial \xi} (\varphi \cdot \bar{1}_\xi) \right) = \frac{\partial \varphi}{\partial x} \cdot \bar{1}_x + \frac{\partial \varphi}{\partial y} \cdot \bar{1}_y + \frac{\partial \varphi}{\partial z} \cdot \bar{1}_z = \operatorname{grad}\varphi . \tag{6}$$

Taking into account (6) the equation (1) in this case can be represented as:

$$\oint_S \varphi \cdot \bar{n} \cdot dS = \oint_V \operatorname{grad}\varphi \cdot dV ,$$

which is the representation of the known theorem about volume integration of gradient of scalar function [1].

4. GREEN THEOREM

Let the function Ψ be the multiplication of scalar function f and derivative of scalar function φ in the direction $\bar{n}$:

$$\Psi = f \cdot \frac{\partial \varphi}{\partial n} . \tag{7}$$

For such a type of function Ψ it is possible to write the following equations:

$$\frac{\partial \Psi}{\partial x} = \left(f \cdot \frac{\partial \varphi}{\partial x} \right) = \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial x} + f \cdot \frac{\partial^2 \varphi}{\partial x^2} ,$$
$$\frac{\partial \Psi}{\partial y} = \left(f \cdot \frac{\partial \varphi}{\partial y} \right) = \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial y} + f \cdot \frac{\partial^2 \varphi}{\partial y^2} ,$$
$$\frac{\partial \Psi}{\partial z} = \left(f \cdot \frac{\partial \varphi}{\partial z} \right) = \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial z} + f \cdot \frac{\partial^2 \varphi}{\partial z^2} ,$$

it results from this that:

$$\sum_{\xi = x,y,z} \frac{\partial}{\partial \xi} \left(f \cdot \frac{\partial \varphi}{\partial \xi} \right) = f \cdot \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) + \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial x} + \tag{8}$$
$$+ \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial z} = f \cdot \Delta\varphi + (\operatorname{grad} f \cdot \operatorname{grad}\varphi) .$$

Taking into account (8) the equation (1) in this case can be written as:

$$\oint_S f\frac{\partial\varphi}{\partial n}\cdot dS=\oint_V[f\cdot\Delta\varphi+(\mathrm{grad}f\cdot\mathrm{grad}\varphi)]\cdot dV,$$

which is the notation of the Green theorem [1-3].

5. VOLUME INTEGRATION OF EXPRESSION OF THE TYPE

Let the function Ψ be defined by the following equation:

$$\Psi=\bar b\cdot(\bar a\cdot\bar n). \tag{9}$$

For such type of the function Ψ it is possible to obtain the following equations:

$$\frac{\partial}{\partial x}\left(\bar b\cdot\left(\bar a\cdot\bar 1_x\right)\right)=\frac{\partial}{\partial x}\left(\bar a\cdot\bar b\right)=\bar b\cdot\frac{\partial a_x}{\partial x}+a_x\cdot\frac{\partial\bar b}{\partial x},$$
$$\frac{\partial}{\partial y}\left(\bar b\cdot\left(\bar a\cdot\bar 1_y\right)\right)=\frac{\partial}{\partial y}\left(\bar a\cdot\bar b\right)=\bar b\cdot\frac{\partial a_y}{\partial y}+a_y\cdot\frac{\partial\bar b}{\partial y},$$
$$\frac{\partial}{\partial z}\left(\bar b\cdot\left(\bar a\cdot\bar 1_z\right)\right)=\frac{\partial}{\partial z}\left(\bar a\cdot\bar b\right)=\bar b\cdot\frac{\partial a_z}{\partial z}+a_z\cdot\frac{\partial\bar b}{\partial z},$$

and taking into account given above correlations it can be obtained that:

$$\sum_{\zeta=x,y,z}\frac{\partial}{\partial\zeta}\psi\left(\bar a,\bar b,\bar 1_\zeta\right)=\bar b\cdot\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right)+ \tag{9}$$
$$+a_x\frac{\partial\bar b}{\partial x}+a_y\frac{\partial\bar b}{\partial y}+a_z\frac{\partial\bar b}{\partial z}=\bar b\cdot\mathrm{div}\bar a+\left(\bar a\cdot\mathrm{grad}\right)\bar b.$$

It results from (9) that the equation (1) in this case can be represented in the form:

$$\oint_S\bar b\left(\bar n\cdot\bar a\right)\cdot dS=\oint_V\left[\left(\bar a\cdot\mathrm{grad}\right)\bar b+\bar b\cdot\mathrm{div}\bar a\right]\cdot dV,$$

which coincides with the known correlation [4].

In conclusion of consideration of equation (1) it should be noted that all types of the above-considered functions Ψ (2), (3), (5), (7), (9) satisfy the property of oddness:

$$\Psi\left(\bar a,\bar b,\bar n\right)=-\Psi\left(\bar a,\bar b,-\bar n\right).$$

It should be expected that the equation (1) allows to obtain new integral equations. But it requires a general proof of equation (1) which can be found as generalization of the known proofs of theorems about volume integration of differential-vectorial operations.

CONCLUSIONS

1. Known theorems about volume integration of differential-vectorial operations are generalized in the form of representation.

2. The necessity of general proof of equation (1) for volume integration of differential-vectorial operations is shown in order to obtain new integral equations.

REFERENCES

Korn G., Korn T., 1968: Mathematical handbook for scientists and engineers. Definitions, theorems and formulas for reference and review. McGraw – Hill Book Company, New York – San Francisko –Toronto – London – Sydney.

Bronshtein I.N., Semendjaev K. A., 1986: Reference book on mathematics for engineers and students. Nauka, Moscow: 544. (in Russian)

Myshkis A. D., 1969: Lectures on higher mathematics. Nauka, Moscow: 640. (in Russian)

Zagirnyak M. V., Branspiz Yu. A., 1996: Calculation of direct magnetic field force on the ferromagnetic body. Proceedings of the XIV-th Symposium "Electromagnetic phenomena in nonlinear circuits" (EPNC'96), Poznan: 16-164.

UOGÓLNIENIE INTEGRALNYCH REPREZENTACJI TWIERDZEŃ

Streszczenie. Wykazano, że znane twierdzenia o integracji objętościowej pewnych operacji różniczkowo-wektorialnych mogą być zapisane jako jedno działanie analityczne.

Słowa kluczowe: integracja objętościowa, integracja powierzchniowa.